# Solved Problems
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# Chapter 1: Probability Theory
# Chapter 2: Random Variable

By:

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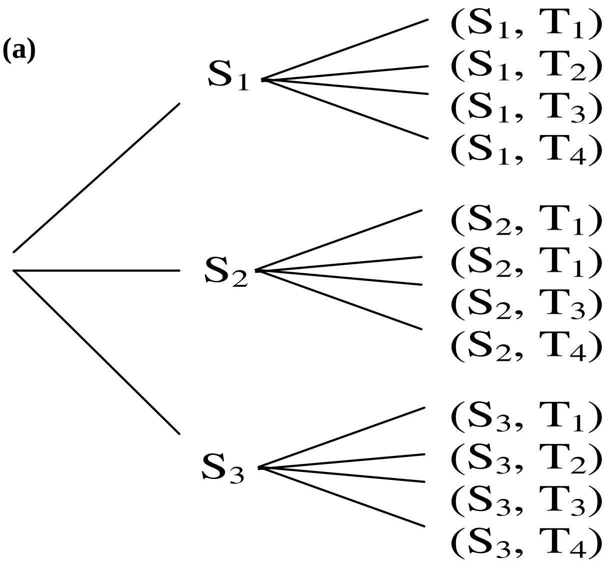
## Problem 1:

(a) A man has 3 shirts and 4 ties. Make a tree diagram to show all possible ways of choosing a shirt and tie.

(b) b) Assuming the man has no preference for specific shirt and tie combinations, and all his shirts and ties are available. What is the probability of each possibility in (a)?

**Solution:**

**(a)**




**(b)** N = 12, P = 1/12.

## Problem 2:

Two dice are thrown . let E be the event that the sum of the dice is odd; let F be the event that at least one of the dice lands on 1; and let G be the event that the sum is 5. Describe the events EF, E $\cup$ F, FG, FG`.

**Solution:**

$EF = \{(1, 2), (1, 4), (1, 6), (2, 1), (4, 1), (6, 1)\}$.

$E \cup F$ occurs if the sum is odd or if at least one of the dice lands on 1. $FG = \{(1, 4), (4, 1)\}$.

$EF^c$ is the event that neither of the dice lands on 1 and the sum is odd. $EFG = FG$.

**Problem 3:**

Sixty percent of the students at a certain school wear neither a ring nor a necklace. Twenty percent wear a ring and 30 percent wear a necklace. If one of the students is chosen randomly, what is the probability that this student is wearing

(a) A ring or necklace
(b) A ring and a necklace

**Solution:**

Let $R$ and $N$ denote the events, respectively, that the student wears a ring and wears a necklace.

(a) $P(R \cup N) = 1 - .6 = .4$

(b) $.4 = P(R \cup N) = P(R) + P(N) - P(RN) = .2 + .3 - P(RN)$
Thus, $P(RN) = .1$

**Problem 4:**

A small community organization consists of 20 families, of which 4 have one child, 8 have two children, 5 have three children, 2 have 4 children, and 1 has five children.

(a) If one of these families is chosen at random, what is the probability it has i children, i = 1,2,3,4,5?
(b) If one of the children is randomly chosen, what is the probability this child comes from a family having i children, i =1,2,3,4,5?
**(c)**

**Solution:**

(a) $p_1 = 4/20, p_2 = 8/20, p_3 = 5/20, p_4 = 2/20, p_5 = 1/20$

(b) There are a total of $4 \cdot 1 + 8 \cdot 2 + 5 \cdot 3 + 2 \cdot 4 + 1 \cdot 5 = 48$ children. Hence,

$$q_1 = 4/48, q_2 = 16/48, q_3 = 15/48, q_4 = 8/48, q_5 = 5/48$$

**Problem 5:**

Five people, designed as A,B,C,D,E, are arranged in linear order. Assuming that each possible order is equally likely, what is the probability that:

(a) There is exactly one person between A and B;
(b) There are exactly 2 people between A and B;
(c) There are 3 people between A and B?

**Solution:**

(a) If *A* is first, then *A* can be in any one of 3 places and *B*'s place is determined, and the others can be arranged in any of 3! ways. As a similar result is true, when *B* is first, we see that the probability in this case is $2 \cdot 3 \cdot 3!/5! = 3/10$

(b) $2 \cdot 2 \cdot 3!/5! = 1/5$

(c) $2 \cdot 3!/5! = 1/10$

**Problem 6:**

A group of 6 men and 6 women is randomly divided into 2 groups of size 6 each. What is the probability that both groups will have the same number of men?

**Solution:**

$$\binom{6}{3}\binom{6}{3}\Big/\binom{12}{6}$$

**Problem 7:**

One chip is selected at random from a box containing five chips numbered 1, 2, 3, 4, 5.

a) What is the probability of selecting a 1?
b) Suppose we are told that the outcome is an odd number . We are given that the outcome is in A. What is the probability of getting a 1?

**Solution:**

a) S = {1, 2, 3, 4, 5} is an equally probable sample space.
Let A = {1}, then P(A) = 1/5

b) B = event of an odd number = {1, 3, 5}, $P(A|B) = P(A \cap B)/P(B) = (1/5)/(3/5) = 1/3$.

**Problem 8:**

Human blood may contain either or both of two antigens, A and B. Blood that contains only the A antigen is called type A, blood that contains only the B antigen is called type B, blood that contains both antigens is called AB, and blood that contains neither is called type O. At a certain blood bank, 35% of the blood donors have type A, 10% have type B, and 5% have type AB.

(a) What is the probability that randomly chosen blood donor id type O?

(b) A recipient with type A blood may safely receive blood from a donor whose blood does not contain the B antigen. What is the probability a randomly chosen blood donor may donate to a recipient with type A blood?

**Solution:**

(a) P(type O) = 1 – P(type A or B or AB)
= 1- (0.35 + 0.1 + 0.05) = 0.5

(b) P( person does not have B antigen) = 1- P(type B or type AB)
= 1- (0.1 + 0.05) = 0.85

**Problem 9:**

Five separate awards (best scholarship, best leadership qualities, and so on) are to be presented to select students from a class of 30. how many different outcomes are possible if:

(a) A student can receive any number of awards;

(b) Each student can receive at most 1 award?

**Solution:**

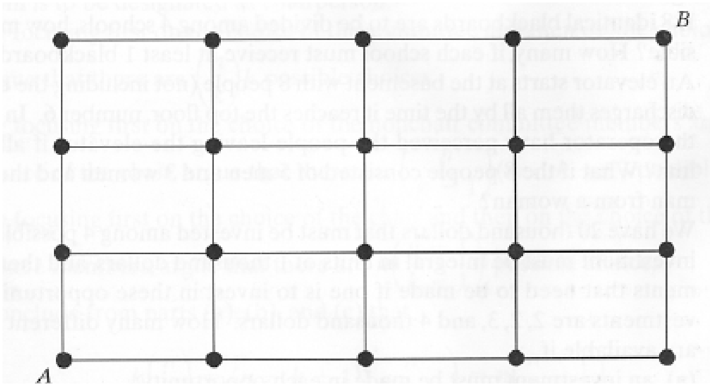
(a) Each scholarship can be awarded to any of the 30 students, so the answer is $30^5$

(b) For the first award we have 30 student to choose from, for the second one we have 29 and so on. This means the answer is 30.29.28.27.26 = 17,100,720

**<u>Problem 10</u>**

Consider the grid of points shown below. Suppose that starting at the point labeled A you can go one step up or one step to the right at each move. This is continues until the point labeled B is reached. How many different paths from A to B are possible?

Hint: Note that to reach B from A you must take 4 steps to the right and 3 steps upward.




**Solution:**

We show the paths with a sequence of r and u instead of right and up, respectively. For example, rruurru is a valid path. All possible sequences of length 7 with 4 r and 3 u equal to $\frac{7!}{3!\,4!}$ .

## Problem 11

**1.** Let $X$ be a random variable with probability density function

$$f(x) = \begin{cases} c(1 - x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

**(a)** What is the value of $c$? **(b)** What is the cumulative distribution function of $X$?

**Solution:**

(a) $c\int_{-1}^{1}(1-x^2)dx = 1 \Rightarrow c = 3/4$

(b) $F(x) = \frac{3}{4}\int_{-1}^{x}(1-x^2)dx = \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right), -1 < x < 1$

## Problem 12

The probability density function of $X$, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \leq 10 \end{cases}$$

**(a)** Find $P\{X > 20\}$.
**(b)** What is the cumulative distribution function of $X$?

## Solution:

(a) $\int_{20}^{\infty}\frac{10}{x^2}dx = \frac{-10}{x}\Big|_{20}^{\infty} = 1/2$ .

(b) $F(y) = \int_{10}^{y}\frac{10}{x^2}dx = 1 - \frac{10}{y}, y > 10.$ $F(y) = 0$ for $y < 10$.

## Problem 13

The density function of $X$ is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = \frac{3}{5}$, find $a$ and $b$.

## Solution:

$$\int_0^1 (a + bx^2)dx = 1 \text{ or } a + \frac{b}{3} = 1$$

$$\int_0^1 x(a + bx^2)dx = \frac{3}{5} \text{ or } \frac{a}{2} + \frac{b}{4} = 3/5. \text{ Hence,}$$

$$a = \frac{3}{5}, \; b = \frac{6}{5}$$

## Problem 14

**2.** A system consisting of one original unit plus a spare can function for a random amount of time $X$. If the density of $X$ is given (in units of months) by

$$f(x) = \begin{cases} Cxe^{-x/2} & x > 0 \\ 0 & x \le 0 \end{cases}$$

what is the probability that the system functions for at least 5 months?

**Solution:**

$$\int xe^{-x/2}dx = -2xe^{-x/2} - 4e^{-x/2}. \text{ Hence,}$$

$$c\int_0^\infty xe^{-x/2}dx = 1 \Rightarrow c = 1/4$$

$$P\{X > 5\} = \frac{1}{4}\int_5^\infty xe^{-x/2}dx = \frac{1}{4}[10e^{-5/2} + 4e^{-5/2}]$$

$$= \frac{14}{4}e^{-5/2}$$

## Problem 15

**3.** Consider the function

$$f(x) = \begin{cases} C(2x - x^3) & 0 < x < \frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$$

Could $f$ be a probability density function? If so, determine $C$. Repeat if $f(x)$ were given by

$$f(x) = \begin{cases} C(2x - x^2) & 0 < x < \frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$$

**Solution:**

No. $f(5/2) < 0$

## Problem 16

**6.** Compute $E[X]$ if $X$ has a density function given by

**(a)** $f(x) = \begin{cases} \frac{1}{4} x e^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$; **(b)** $f(x) = \begin{cases} c(1 - x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$;

**(c)** $f(x) = \begin{cases} \dfrac{5}{x^2} & x > 5 \\ 0 & x \le 5 \end{cases}$.

**Solution:**

(a) $E[X] = \dfrac{1}{4}\int_0^\infty x^2 e^{-x/2} dx = 2\int_0^\infty y^2 e^{-y} dx = 2\Gamma(3) = 4$

(b) By symmetry of $f(x)$ about $x = 0$, $E[X] = 0$

(c) $E[X] = \int_5^\infty \dfrac{5}{x} dx = \infty$